Int J Theor Phys (2009) 48: 3173–3187
DOI 10.1007/s10773-009-0118-7

Antiparticle Contribution in the Cross Ladder Diagram for Bethe-Salpeter Equation in the Light-Front

J.H.O. Sales · A.T. Suzuki

Received: 30 April 2009 / Accepted: 29 July 2009 / Published online: 11 August 2009
© Springer Science+Business Media, LLC 2009

Abstract We construct the homogeneous integral equation for the vertex of the bound state in the light front with the kernel approximated to order g^4. We will truncate the hierarchical equations from Green functions to construct dynamical equations for the two boson bound state exchanging interacting intermediate σ bosons and including pair creation process contributing to the crossed ladder diagram.

Keywords Bethe-Salpeter equation · Cross-ladder diagram · Light-front

1 Introduction

In relativistic quantum field theories the Bethe-Salpeter equation describes a system of two interacting particles [1–3]. In general, for practical applications, to study the bound state of two particles, its kernel is truncated in its lowest order. Today, it is already possible to solve the Bethe-Salpeter equation including the "ladder" diagram and "crossed ladder" diagrams in its kernel [4]. Also, recently, the Bethe-Salpeter equation in the "ladder" approximation has been solved in Minkowski space [5]. However up to the time of the publication of J.H.O. Sales work [6–8] there was no detailed comparison between the results in the four dimensional calculations and in the light front, with the kernel including the propagation of intermediate states of four bodies. The objective was to study the effect of the components of the Fock space with particle number greater than three, going beyond the previously done works in the area.

The concentrated effort to use the tridimensional reduction in light front coordinates is justifiable, since there is hope that using light front quantization would make it possible to better understand several aspects of low energy quantum chromodynamics (QCD) [9]. In the past, the concept of wave function in the light front was applied in the context of nuclear

J.H.O. Sales · A.T. Suzuki (✉)
Instituto de Física Teórica, UNESP, Rua Dr. Bento Teobaldo Ferraz, 271–Bloco II,
01140-070 São Paulo, SP, Brazil
e-mail: suzuki@ift.unesp.br

physics to describe the deuteron [10–14] and the discussion of its properties in the light front extends up to now [15].

The representation of the Bethe-Salpeter equation in the infinite momentum frame was studied by Weinberg [16], which corresponds to truncation of the Fock space in the light front up to the intermediate state of three particles. In this approximation numerical results in different contexts have been obtained, such as in bosonic models [17–19] and in fermionic ones [20, 21]. However, an explicit systematic expansion for the Bethe-Salpeter equation in the light front was still lacking in the literature. Although in the past there had been attempts to include intermediate states of more particles but in an incomplete way [22].

In this work we explore the concept of covariant quantum propagator written down in terms of light front coordinates and obtain the propagator and Green's function in the light-front for a time interval x^+ where $x^+ = t + z$, is the light-front "time" . In principle, this is equivalent to the canonical quantization in the light front [9, 23]. Kogut and Soper [24] also makes use of this way of constructing quantities in the light front: starting with 4-dimensional amplitudes or equations, they integrate over $k^- = k^0 - k^3$, which plays the role of "energy" and corresponds to processes described by amplitudes or equations in "time" x^+. With this, the relative time between particles disappear and only the global propagation of intermediate state is allowed. The global propagation of the intermediate state is the "time" translation of the physical system between two instants x^+ and x'^+.

Also, in the Minkowski space we have the relation between rest mass and energy for a free particle given by

$$k^\mu k_\mu = m^2. \tag{1}$$

Using the light-front coordinates, we have

$$k^- = \frac{\vec{k}_\perp^2 + m^2}{2k^+}. \tag{2}$$

Note that the energy of a free particle is given by $k^0 = \pm\sqrt{m^2 + \mathbf{k^2}}$, which shows us a quadratic dependence between k^0 and $\mathbf{k}$ in contrast to the linear dependence between $(k^+)^{-1}$ and k^- as seen in (2) [28].

This last result is very important. It means that particles propagating forward in the null-plane time have positive k^+. As we will see an energy integration in the null-plane arises from projecting the propagation of the physical system in time x^+, which implies that in the Bethe-Salpether equation and in the perturbative expansion of the propagator for two particles between two intants of time x^+, non-vanishing contributions will appear only for $k^+ > 0$. In principle, therefore we have only propagation of particles. For the ladder diagram this is true. In spite of that we will show in the next sections that in the case of crossed ladder diagrams, besides the contribution from particles we have a non vanishing contribution coming from the anti-particle region, $k^+ < 0$. This is a surprising and unexpected result, since integrations in the light-front coordinates usually are done taking into account only the particle region.

Cooke and Miller [25] have done a study of the Wick-Cutkosky model in the light front and observed the possibility of the z crossed diagram contribution to that model; yet it passed unnoticed to them that that diagram in fact involves the antiparticle region.

2 Free Boson

The propagation of a free particle with spin zero in four dimensional space-time is represented by the covariant Feynman propagator

$$S(x^\mu) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik^\mu x_\mu}}{k^2 - m^2 + i\varepsilon}, \tag{3}$$

where the coordinate x^0 represents the time and k^0 the energy. We are going to calculate this propagation in the light front, that is, for times x^+.

A point in the four-dimensional space-time is defined by the set of numbers (x^0, x^1, x^2, x^3), where x^0 is the time coordinate and $\mathbf{x} = (x^1, x^2, x^3)$ is the three-dimensional vector with space coordinates $x^1 = x$, $x^2 = y$ and $x^3 = z$. Observe that we adopt here the usual convention to take the speed of light as $c = 1$.

In the light front, time and space coordinates are mixed up and we define the new coordinates as follows:

$$x^+ = x^0 + x^3, \qquad x^- = x^0 - x^3 \quad \text{and} \quad \vec{x}^\perp = x^1\vec{i} + x^2\vec{j}, \tag{4}$$

where $\vec{i}$ and $\vec{j}$ are unit vectors in the direction of x and y coordinates respectively.

The null plane is defined by $x^+ = 0$, that is, this condition defines a hyperplane that is tangent to the light-cone, the reason why many authors call the hypersurface simply by light-cone.

The initial boundary conditions for the dynamics in the light front are defined on this hyperplane. The axis x^+ is perpendicular to the plane $x^+ = 0$. Therefore a displacement of such hyperplane for $x^+ > 0$ is analogous to the displacement of a plane in $t = 0$ to $t > 0$ of the four-dimensional space-time. With this analogy, we recognize x^+ as the time in the null plane.

We make the projection of the propagator for a boson in time associated to the null plane rewriting the coordinates in terms of time coordinate x^+ and the position coordinates (x^- and $\vec{x}_\perp$). With these, the momenta are given by k^-, k^+ and $\vec{k}_\perp$, and therefore we have

$$S(x^+) = \frac{1}{2} \int \frac{dk_1^-}{(2\pi)} \frac{i e^{\frac{-i}{2} k_1^- x^+}}{k_1^+ (k_1^- - \frac{k_{1\perp}^2 + m^2 - i\varepsilon}{k_1^+})}. \tag{5}$$

The Jacobian of the transformation k^0, $\vec{k} \to k^-, k^+, \vec{k}_\perp$ is equal to $\frac{1}{2}$ and k^+, $k_\perp$ are momentum operators.

Evaluating the Fourier transform, we obtain

$$\widetilde{S}(k^-) = \int dx^+ e^{\frac{i}{2} k^- x^+} S(x^+), \tag{6}$$

where we have used

$$\delta\left(\frac{k^- - k_1^-}{2}\right) = \frac{1}{2\pi} \int dx^+ e^{\frac{i}{2}(k^- - k_1^-) x^+}, \tag{7}$$

and the property of Dirac's delta "function" $\delta(ax) = \frac{1}{a}\delta(x)$ and we get

$$\widetilde{S}(k^-) = \frac{i}{k^+(k^- - \frac{k_\perp^2 + m^2 - i\varepsilon}{k^+})}, \tag{8}$$

which describes the propagation of a particle forward to the future and of an antiparticle backwards to the past. This can be observed by the denominator which hints us that for $x^+ > 0$ and $k^+ > 0$ we have the particle propagating forward in time of the null plane. On the other hand, for $x^+ < 0$ and $k^+ < 0$ we have an antiparticle propagating backwards in time.

The Green function in the light front $G(x^+)$ acting in the Fock space is defined as the probability amplitude of the transition from the initial state in the Fock space $|i\rangle$ to the final state $|f\rangle$. Its Fourier transform is sometimes called resolvent for a given Hamiltonian [26], however, here we call simply Fourier transform for the Green function or even Green function itself.

In the case of a free boson, the Green function for the propagation of a particle is defined by the operator

$$G_0^{(1p)}(k^-) = \frac{\theta(k^+)}{k^- - k^-_{on} + i\varepsilon}; \tag{9}$$

where $k^-_{on} = \frac{k_\perp^2 + m^2}{k^+}$ is the energy of the particle. For the antiparticle propagation, we have:

$$G_0^{(1a)}(k^-) = \frac{\theta(-k^+)}{k^- - k^-_{on} - i\varepsilon}. \tag{10}$$

We can see that the difference between the Green functions in (9) and (10) for the propagator in the light front is the absence of the imaginary (complex) number i and of the factor of phase space k^+ which appears in (8).

The operator defined by (9) is the Green function of

$$(k^- - k^-_{on})(G_0^{(1p)}(k^-) + G_0^{(1a)}(k^-)) = 1. \tag{11}$$

The Feynman propagator is then rewritten as:

$$S(k^\mu) = \frac{i}{k^+} G_0^{(1p)}(k^-) - \frac{i}{|k^+|} G_0^{(1a)}(k^-) = \frac{i}{k^+(k^- - \frac{k_\perp^2 + m^2 - i\varepsilon}{k^+})}. \tag{12}$$

3 Green Function of Two Bosons

Our aim in this chapter is to study the two body Green function in the "ladder" approximation for the dynamics defined in the light front. Within this treatment, we are not going to deal with perturbative corrections that can be decomposed into one body problem.

Our interest is to define in the light front, the interaction between two bodies mediated by the interchange of a particle and obtain the correction to the two body Green function originated in this interaction.

For this purpose, we use a bosonic model for which the interaction Lagrangian is defined as:

$$\mathcal{L}_I = g\phi_1^*\phi_1\sigma + g\phi_2^*\phi_2\sigma, \tag{13}$$

where the bosons ϕ_1 and ϕ_2 have equal mass m and the intermediate boson, σ, has the mass m_σ. The coupling constant is g.

Taking from Dirac's idea [27] of representing the dynamics of a quantum system in the light front in time $x^+ = t + z$, we derive in this chapter the two body Green function or covariant propagator which describes the evolution of the system from a hypersurface $x^+ =$ constant to another one. The Green function in the light front is the probability amplitude for a initial state in $x^+ = 0$ evolving to a final state in $x^+ > 0$, where the evolution operator is defined by the Hamiltonian in the light front [28]. Sometimes we call the resolvent $(Z - H)^{-1}$ as the Green function, too [9].

The two body Green function in the light front includes the propagation of intermediate states with any number of particles.

We start our discussion evaluating the second order correction to the coupling constant associated with the propagator. We define the matrix element for the interaction and so we obtain the correction to the Green function in the light front. Then we evaluate the correction to the Green function to the fourth order in the coupling constant, where we use the technique of factorizing the energy denominators, which is important to identify the global propagation of four bodies after the integration in the energies k^-. We discuss the generalization of this technique.

We show how to get the non perturbative Green function from the set of hierarchical equations for the Green function in the "ladder" approximation. This corresponds to the truncation of the Fock space in the light front, such that, only states with two bosons ϕ_1 and ϕ_2 are allowed, with no restriction as to the number of intermediate bosons σ. We discuss how to build a systematic approximation to the kernel of the integral equation for the two body Green function as a function of the number of particles in the intermediate Fock state and the power in the coupling constant. A consistent truncation can be carried out and in the lowest order, this brings to the Weinberg's equation for the bound state [16].

This diagram brings about a new feature which was not present in the previous diagram just considered. Here we come across not only with all those diagrams that involve the propagation of information to future times, but note one particular diagram that bears pair production, i.e., there is one diagram which has intrinsic propagation of information to the past, thus mingling the two sectorized Fock spaces of solutions. This diagram, as far as we know, has not been considered in the literature before.

Explicitly this correction to the propagator is written down in terms of the one boson propagators and is given by the following equation

$$\Delta S_\times(x^+) = (ig)^4 \int d\bar{x}_1^+ d\bar{x}_2^+ d\bar{x}_3^+ d\bar{x}_4^+ S_3(x^+ - \bar{x}_2^+) S_\sigma(\bar{x}_2^+ - \bar{x}_3^+) S_2(\bar{x}_2^+ - \bar{x}_1^+) S_6(x^+ - \bar{x}_4^+)$$
$$\times S_\sigma(\bar{x}_4^+ - \bar{x}_1^+) S_5(\bar{x}_4^+ - \bar{x}_3^+) \times S_1(\bar{x}_1^+) S_4(\bar{x}_3^+), \tag{14}$$

and after Fourier transform we have

$$\Delta \widetilde{S}_\times(K^-) = \frac{(ig)^4 i^8}{2^3(2\pi)^3} \int dk^- dp^- dq^- \frac{dp^+ d^2 p_\perp}{k^+ p^+ (k-p)^+ (p-q)^+ (K-k-q+p)^+}$$
$$\times \frac{1}{(K-q)^+ (q-p)^+ (k-p)^+}$$
$$\times \frac{1}{(k^- - \frac{k_\perp^2 + m^2 - i\varepsilon}{k^+})} \frac{1}{(p^- - \frac{k_\perp^2 + m^2 - i\varepsilon}{p^+})}$$
$$\times \frac{1}{(q^- - \frac{k_\perp^2 + m^2 - i\varepsilon}{q^+})} \frac{1}{(K^- - k^- - \frac{(K-k)_\perp^2 + m^2 - i\varepsilon}{(K-k)^+})}$$

$$\times \frac{1}{(K^- - q^- - \frac{(K-q)_\perp^2 + m^2 - i\varepsilon}{(K-q)^+})} \frac{1}{(k^- - p^- - \frac{(k-p)_\perp^2 + m_\sigma^2 - i\varepsilon}{(k-p)^+})}$$
$$\times \frac{1}{(K^- - k^- - q^- + p^- - \frac{(K-k-q+p)_\perp^2 + m^2 - i\varepsilon}{(K-k-q+p)^+})}$$
$$\times \frac{1}{(q^- - p^- - \frac{(q-p)_\perp^2 + m_\sigma^2 - i\varepsilon}{(q-p)^+})}. \tag{15}$$

For $K^+ > 0$, the regions of integration in p^+ which define the position of poles in the complex p^- are:

(a) $0 < q^+ < p^+ < k^+ < K^+$,
(b) $0 < k^+ < p^+ < q^+ < K^+$,
(c) $0 < p^+ < q^+ < k^+ < K^+$,
(d) $0 < p^+ < k^+ < q^+ < K^+$,
(e) $0 < k^+ < q^+ < p^+ < K^+$,
(f) $0 < q^+ < k^+ < p^+ < K^+$.

For regions (c) and (d) we use the method of partial fractioning twice to integrate in p^- [7, 8]; for (a), (b) , (e) and (f) this is not necessary; for regions (e) and (f) the integration in p^- vanishes. The diagrams for each of these regions are shown in Fig. 1.

After performing the analytic integrations in k^-, p^- and q^- we have

$$\Delta\widetilde{S}_\times(K^-) = \Delta\widetilde{S}_\times^a(K^-) + \Delta\widetilde{S}_\times^b(K^-) + \Delta\widetilde{S}_\times^c(K^-) + \Delta\widetilde{S}_\times^d(K^-) \tag{16}$$

where

$$\Delta\widetilde{S}_\times^a(K^-) = (ig)^4 \int \frac{i dp^+ d^2 p_\perp \theta(k^+ - p^+)\theta(p^+ - q^+)}{2k^+(K-k)^+(K^- - \frac{k_\perp^2 + m^2 - i\varepsilon}{k^+} - \frac{(K-k)_\perp^2 + m^2 - i\varepsilon}{(K-k)^+})}$$

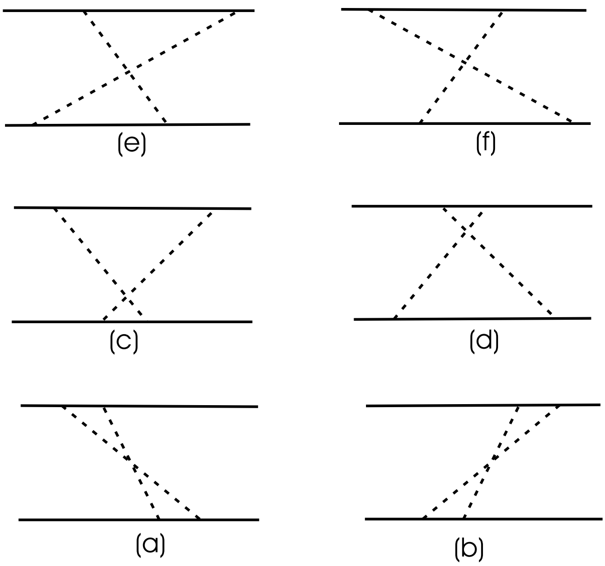

Fig. 1 Crossed ladder diagrams time-ordered in $x^+ > 0$

$$
\begin{aligned}
&\times \frac{1}{2p^+(k-p)^+(p-q)^+(K-k-q+p)^+} \\
&\times \frac{i}{(K^- - \frac{p_\perp^2+m^2-i\varepsilon}{p^+} - \frac{(K-k)_\perp^2+m^2-i\varepsilon}{(K-k)^+} - \frac{(k-p)_\perp^2+m^2-i\varepsilon}{(k-p)^+})} \\
&\times \frac{i}{(K^- - \frac{(p-q)_\perp^2+m_\sigma^2-i\varepsilon}{(p-q)^+} - \frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{(k-p)^+} - \frac{(K-k)_\perp^2+m^2-i\varepsilon}{(K-k)^+} - \frac{q_\perp^2+m^2-i\varepsilon}{q^+})} \\
&\times \frac{i}{(K^- - \frac{q_\perp^2+m^2-i\varepsilon}{q^+} - \frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{(k-p)^+} - \frac{(K-k-q+p)_\perp^2+m^2-i\varepsilon}{(K-k-q+p)^+})} \\
&\times \frac{i}{2q^+(K-q)^+(K^- - \frac{q_\perp^2+m^2-i\varepsilon}{q^+} - \frac{(K-q)_\perp^2+m^2-i\varepsilon}{(K-q)^+})},
\end{aligned}
\tag{17}
$$

and

$$
\Delta\widetilde{S}_\times^b(K^-) = \Delta\widetilde{S}_\times^a(K^-)[k \leftrightarrow q],
\tag{18}
$$

shown in Figs. 1(a) and (b) respectively.

Regions (c) and (d) contribute to the propagator correction as

$$
\begin{aligned}
\Delta\widetilde{S}_\times^c(K^-) = (ig)^4 \int &\frac{i dp^+ d^2 p_\perp \theta(q^+-p^+)\theta(k^+-p^+)}{2k^+(K-k)^+(K^- - \frac{k_\perp^2+m^2-i\varepsilon}{k^+} - \frac{(K-k)_\perp^2+m^2-i\varepsilon}{(K-k)^+})} \\
&\times \frac{1}{2p^+(k-p)^+(p-q)^+(K-k-q+p)^+} \\
&\times \frac{i}{(K^- - \frac{p_\perp^2+m^2-i\varepsilon}{p^+} - \frac{(q-p)_\perp^2+m_\sigma^2-i\varepsilon}{(q-p)^+} - \frac{(K-k-q+p)_\perp^2+m^2-i\varepsilon}{(K-k-q+p)^+} - \frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{(k-p)^+})} \\
&\times \widetilde{S'}_\times^c \frac{i}{2q^+(K-q)^+(K^- - \frac{q_\perp^2+m^2-i\varepsilon}{q^+} - \frac{(K-q)_\perp^2+m^2-i\varepsilon}{(K-q)^+})},
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\widetilde{S'}_\times^c = &\frac{i}{(K^- - \frac{q_\perp^2+m^2-i\varepsilon}{q^+} - \frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{(k-p)^+} - \frac{(K-k-q+p)_\perp^2+m^2-i\varepsilon}{(K-k-q+p)^+})} \\
&\times \frac{i}{(K^- - \frac{k_\perp^2+m^2-i\varepsilon}{k^+} - \frac{(K-k-q+p)_\perp^2+m_\sigma^2-i\varepsilon}{(K-k-q+p)^+} - \frac{(q-p)_\perp^2+m^2-i\varepsilon}{(q-p)^+})} \\
&\times \frac{i}{(K^- - \frac{k_\perp^2+m^2-i\varepsilon}{k^+} - \frac{(q-p)_\perp^2+m_\sigma^2-i\varepsilon}{(q-p)^+} - \frac{(K-k-q+p)_\perp^2+m^2-i\varepsilon}{(K-k-q+p)^+})} \\
&\times \frac{i}{(K^- - \frac{p_\perp^2+m^2-i\varepsilon}{p^+} - \frac{(q-p)_\perp^2+m_\sigma^2-i\varepsilon}{(q-p)^+} - \frac{(K-q)_\perp^2+m^2-i\varepsilon}{(K-q)^+})}.
\end{aligned}
\tag{20}
$$

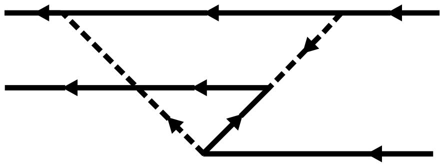

Fig. 2 Pair creation process contributing to the crossed ladder diagram

The perturbative correction to the two boson propagator in (19) is represented by diagrams indicated in Fig. 1(c). The correction represented by diagrams in Fig. 1(d) is given by:

$$\Delta\widetilde{S}^{d}_{\times}(K^-)=\Delta\widetilde{S}^{c}_{\times}(K^-)[k\leftrightarrow K-k,\, p\leftrightarrow K-k-q+p,\, q\leftrightarrow K-q]. \tag{21}$$

4 Antiparticle Contribution

Next we deduce the antiparticle contribution to the crossed ladder diagram. This contribution happens for $p^+<0$ and $K^+-k^+-q^++p^+<0$. Let us analyse the first case, $p^+<0$.

The region for the "+" component momentum that allows the pole positioned in both hemispheres of the complex p^- plane, and therefore giving non-vanishing residue, are $-k^+<p^+<0$ and $|p^+|+k^++q^+<K^+$. So, the result for the momentum integration in "−" component for $0<q^+<K^+$ and $0<k^+<K^+$ which correspond to the non-vanishing results for integrations in k^- and q^- for $-k^+<p^+<0$, is given by the diagram depicted in Fig. 2 and the result is given in:

$$\begin{aligned}
\Delta\widetilde{S}^{ant}_{\times}(K^-)&=\frac{(ig)^4}{8}\int dp^+d^2p_\perp\frac{\theta(k^++|p^+|)\theta(q^++|p^+|)}{k^+|p^+|q^+(|p|+k)^+(q+|p|)^+}\\
&\quad\times\frac{\theta(K^+-|p^+|-q^+-k^+)}{(K-k)^+(K-q)^+(K-|p|-q-k)^+}\\
&\quad\times\frac{i}{(K^--K_0^-)}\frac{i}{(K^--Q_0^-)}\frac{i}{(K^--T^-)}\\
&\quad\times\frac{i}{(K^--J_a^-)}\frac{i}{(K^--T'^-)}\\
&\quad+[k\leftrightarrow q]+[k\to K-k,q\to K-q]\\
&\quad+[k\to K-q,q\to K-k],
\end{aligned}\tag{22}$$

where

$$\begin{aligned}
T^-&=\frac{(p-q)^2_\perp+m^2_\sigma}{q^++|p^+|}+\frac{(K+p-q-k)^2_\perp+m^2}{K^+-k^+-q^+-|p^+|}\\
&\quad+\frac{k^2_\perp+m^2}{k^+},\\
J_a^-&=\frac{q^2_\perp+m^2}{q^+}+\frac{(K-k-q+p)^2_\perp+m^2}{K^+-k^+-q^+-|p^+|}\\
&\quad+\frac{p^2_\perp+m^2}{|p^+|}+\frac{k^2_\perp+m^2}{k^+},
\end{aligned}\tag{23}$$

$$T'^{-} = \frac{(q-p)_\perp^2 + m_\sigma^2}{q^+ + |p^+|} + \frac{(K-k-w+p)_\perp^2 + m^2}{K^+ - k^+ - q^+ - |p^+|} + \frac{q_\perp^2 + m^2}{q^+}.$$

The four body propagator, J_a^-, (subindex a for antiparticle) has a propagation to past in the null-plane of an antiparticle with $p^+ < 0$. At instant $\bar{x}_2^+ > 0$ the pair particle-antiparticle is produced by the σ intermediate boson, then the antiparticle encounters a particle of momentum $k^+ > 0$, and is annihilated and the production of a σ boson with momentum $|p^+| + k^+ > 0$ which continues to propagate into the future of the null-plane.

5 Hierarchical Equations

In general, the Green function in the light front for a system of two bodies could be obtained from the solution for the covariant Bethe-Salpeter equation which has all the two body irreducible diagrams in the kernel and self-energy corrections to the intermediate propagators of the ϕ_1 and ϕ_2 bosons. We can easily obtain the two boson Green function in the light front, without including self-energy corrections to the intermediate bosons, that is, closed loops for the bosons ϕ_1 and ϕ_2 and crossed diagrams, as a solution to the following set of hierarchical equations:

$$\begin{aligned}
G^{(2)}(K^-) &= G_0^{(2)}(K^-) + G_0^{(2)}(K^-)VG^{(3)}(K^-)VG^{(2)}(K^-),\\
G^{(3)}(K^-) &= G_0^{(3)}(K^-) + G_0^{(3)}(K^-)VG^{(4)}(K^-)VG^{(3)}(K^-),\\
G^{(4)}(K^-) &= G_0^{(4)}(K^-) + G_0^{(4)}(K^-)VG^{(5)}(K^-)VG^{(4)}(K^-),\\
&\dots\\
G^{(N)}(K^-) &= G_0^{(N)}(K^-) + G_0^{(N)}(K^-)VG^{(N+1)}(K^-)VG^{(N)}(K^-),\\
&\dots
\end{aligned} \tag{24}$$

The set of equations above, (24), include, in particular, the "ladder" approximation for the covariant Bethe-Salpeter equation. The hierarchical equations, (24), correspond to a truncation in the Fock space in the light front, so that only states with two particles ϕ_1 and ϕ_2 without restriction in the number of σ bosons are permitted in the intermediate states. The free propagation of these states is represented by the Green function $G_0^{(N)}(K^-)$, where the number of σ bosons is $N-2$. Equations (24) do not, however, include the totality of crossed "ladder" diagrams. For example, the intermediate propagation in the light front of a state of one ϕ_1 boson, two ϕ_2 bosons and one ϕ_2 antiboson (four body Fock state) are not included in the proposed hierarchical equations. In order to get the two body propagator in the light front time in the "ladder" approximation, we shall restrict ourselves to the hierarchy of (24).

6 Bethe-Salpeter Equation in the Light Front

In order to build the integral equation for the vertex of the bound state of two interacting bosons in the light front, we need to introduce the concept of "vertex" associated with a wave function of a bound state. To make our discussion more didatic, we introduce the concept of "vertex" through Schrödinger's equation for the two body bound state, which for the center of mass is given by

$$\left(\frac{p^2}{2m}+V\right)|\Psi_B\rangle = -E_B|\Psi_B\rangle, \tag{25}$$

where $\vec{p}$ is the relative momentum of the two particles with same mass m, V is the potential and $-E_B$ is the bounding energy. Solving (25), we have that

$$|\Psi_B\rangle = \frac{1}{(-E_B-\frac{p^2}{m})}V|\Psi_B\rangle, \tag{26}$$

where the vertex is given by $V|\Psi_B\rangle = |\Gamma_B\rangle$.

Our interest is to define the vertex of the bound state of two bosons using it later on in the hierarchical equations of the Green function (24), from which we get the non perturbative equation for the propagator of two bosons:

$$G^{(2)} = G_0^{(2)} + G_0^{(2)}VG^{(3)}VG^{(2)}, \tag{27}$$

which allows, in principle, the appearance of bound states.

Close to the region of bound state energy the Green function has a pole:

$$\lim_{K^-\to K_B^-} G^{(2)}(K^-) = \frac{|\psi_B\rangle\langle\psi_B|}{K^- - K_B^-}, \tag{28}$$

where $|\psi_B\rangle$ is the wave function of the bound state.

So, introducing (28) into (27) and taking the limit $K^-\to K_B^-$, we have

$$|\Psi_B\rangle = G_0^{(2)}(K_B^-)VG^{(3)}(K_B^-)V|\Psi_B\rangle, \tag{29}$$

with the kernel defined by the hierarchical equations (24).

This can also be written as an eigenvalue equation for a mass squared operator

$$[M_0^2 + K^+VG^{(3)}(K_B^-)V]|\Psi_B\rangle = (M_2)^2|\Psi_B\rangle, \tag{30}$$

where $(M_2)^2 = K^+K_B^- - K_\perp^2$, $M_0^2 = K^+K_{(2)on}^- - K_\perp^2$ and $K_{(2)on}^-$ is defined in

$$K_{(2)on}^- = \frac{k_{1\perp}^2+m^2}{k_1^+} + \frac{k_{2\perp}^2+m^2}{k_2^+}.$$

Equation (29) is the homogeneous Bethe-Salpeter equation projected onto the light front, and the vertex is defined by,

$$|\Gamma_B\rangle = (G_0^{(2)}(K_B^-))^{-1}|\Psi_B\rangle. \tag{31}$$

7 Ladder Approximation in $O(g^2)$

To obtain the homogeneous integral equation for the vertex up to g^2 order, multiply (29) by $(G_0^{(2)})^{-1}$ on both sides and using the property $G_0^{(2)}(G_0^{(2)})^{-1}=1$, we have:

$$|\Gamma_B\rangle = VG_0^{(3)}VG_0^{(2)}|\Gamma_B\rangle. \tag{32}$$

In the basis of kinematical momenta and defining the momentum fractions $x = \frac{k^+}{K^+}$, and $y = \frac{q^+}{K^+}$, we have

$$\Gamma_B(\vec{q}_\perp, y) = \int \frac{dx d^2 k_\perp}{x(1-x)} \frac{K^{(3)}(\vec{q}_\perp, y; \vec{k}_\perp, x)}{M_B^2 - M_0^2} \Gamma_B(\vec{k}_\perp, x), \tag{33}$$

where the free mass of the two boson system, in the center of mass $\vec{K}_\perp = 0$, is given by

$$M_0^2 = K^+ K^-_{(2)on} - K_\perp^2 = \frac{k_\perp^2 + m^2}{x(1-x)}, \tag{34}$$

and

$$\begin{aligned} K^{(3)}(\vec{q}_\perp, y; \vec{k}_\perp, x) = {} & \frac{g^2}{16\pi^3} \frac{\theta(x-y)}{(x-y)} \\ & \times \frac{1}{(M_B^2 - \frac{q_\perp^2 + m^2}{y} - \frac{k_\perp^2 + m^2}{1-x} - \frac{(q-k)_\perp^2 + m_\sigma^2}{x-y})} \\ & + [k \leftrightarrow q], \end{aligned} \tag{35}$$

with $M_B^2 = K_B^+ K_B^-$, in the center of mass where $\vec{K}_\perp = 0$. We draw attention to the fact that $|\Gamma_B\rangle$ and $\Gamma_B(\vec{q}_\perp, y)$ are related to each other by a phase space factor, such that $\Gamma_B(\vec{q}_\perp, y) = \sqrt{q^+(K^+ - q^+)} \langle \vec{q}_\perp, q^+ . | \Gamma_B \rangle$.

Equation (33) is the Bethe-Salpeter equation in the null plane in the "ladder" approximation with the interaction calculated in second order in the coupling constant. This is also the same as the equation obtained by Weinberg [16].

8 Ladder Approximation in $O(g^4)$

We build the homogeneous integral equation for the vertex of the bound state in the light front starting from (24) which has the approximate kernel to order g^4, whose non perturbative solution results in the Green function of two bosons:

$$G_{g^4}^{(2)}(K^-) = G_0^{(2)}(K^-) + G_0^{(2)}(K^-) V G^{(3)}(K^-) V G_{g^4}^{(2)}(K^-),$$

where

$$\begin{aligned} G^{(3)}(K^-) &\cong G_0^{(3)}(K^-) + \Delta G_{g^2}^{(3)}(K^-) \\ &= G_0^{(3)}(K^-) + G_0^{(3)}(K^-) V G_0^{(4)}(K^-) V G_0^{(3)}(K^-). \end{aligned}$$

Truncate now $G^{(3)}$ to order g^2 and $G^{(4)}$ up to g^0, including in this way, the intermediate propagations of three and four bodies in the kernel of the integral equation for $G_{g^4}^{(2)}$.

The integral equation for the vertex of the bound state through the fourth order is built in an analogous form as done previously, where the vertex is given now by

$$|\Gamma_B\rangle = V(G_0^{(3)}(K^-) + \Delta G_{g^2}^{(3)}(K^-)) V G_0^{(2)} |\Gamma_B\rangle. \tag{36}$$

We define the incoming particles in the diagram by the fraction of momentum, $x = \frac{k^+}{K^+}$, and $\vec{k}_\perp$, where as the outgoing one by $y = \frac{q^+}{K^+}$ and $\vec{q}_\perp$. The momenta flowing in the loop which generates the intermediate state of four bodies we define as $z = \frac{p^+}{K^+}$ and $\vec{p}_\perp$, so that we have:

$$\Gamma_B(\vec{q}_\perp, y) = \int \frac{dx d^2 k_\perp}{x(1-x)} \times \frac{K^{(3)}(\vec{q}_\perp, y; \vec{k}_\perp, x) + K^{(4)}(\vec{q}_\perp, y; \vec{k}_\perp, x)}{M_B^2 - M_0^2} \Gamma_B(k_\perp, x), \tag{37}$$

with

$$\begin{aligned} K^{(4)}(\vec{q}_\perp, y; \vec{k}_\perp, x) = {}& \left(\frac{g^2}{16\pi^3}\right)^2 \int \frac{d^2 p_\perp dz}{z(z-x)(y-z)(1-z)} \\ & \times \frac{\theta(y-z)\theta(z-x)}{(M_B^2 - \frac{k_\perp^2 + m^2}{x} - \frac{p_\perp^2 + m^2}{1-z} - \frac{(p-k)_\perp^2 + m_\sigma^2}{z-x})} \\ & \times \frac{1}{(M_B^2 - \frac{p_\perp^2 + m^2}{z} - \frac{q_\perp^2 + m^2}{1-y} - \frac{(q-p)_\perp^2 + m_\sigma^2}{y-z})} \\ & \times \frac{1}{(M_B^2 - \frac{k_\perp^2 + m^2}{x} - \frac{q_\perp^2 + m^2}{1-y} - \frac{(q-p)_\perp^2 + m_\sigma^2}{y-z} + \frac{(p-k)_\perp^2 + m_\sigma^2}{z-x})} \\ & + [x \leftrightarrow y, k_\perp \leftrightarrow q_\perp]. \end{aligned} \tag{38}$$

The vertex $|\Gamma_B\rangle$ and $\Gamma_B(\vec{q}_\perp, y)$ are related like:

$$\langle \vec{q}_\perp, q^+ . | \Gamma_B \rangle = \sqrt{q^+(K^+ - q^+)} \Gamma_B(\vec{q}_\perp, y).$$

Equation (37) is the Bethe-Salpeter equation in the light front with the kernel to the fourth order in the coupling constant in the "ladder" approximation. It is to be noted that we were able to construct the Bethe-Salpeter equation in the "ladder" approximation from the Green function in the light front, and if we need to go to higher orders in g it is only necessary to add terms that include the propagation of N bodies in the non perturbative hierarchical equations (24).

9 Cross Ladder

The construction of the integral equation for the vertex follows the same procedure as before; however now we have to add in (36) the Green function for the crossed diagram (crossed ladder plus antiparticle contribution)

$$\begin{aligned} |\Psi_B\rangle = {}& G_0^{(2)}(K_B^-) V \{ G_0^{(3)}(K_B^-) \\ & + G_0^{(3)}(K_B^-) V [G^{(4)}(K_B^-) + G_\times^{(4)}(K_B^-) + G_\times^{(4)anti}(K_B^-)] V G_0^{(3)}(K_B^-) \} V |\Psi_B\rangle. \end{aligned} \tag{39}$$

The vertex for the bound state satisfies the following homogeneous integral equation now

$$\Gamma_B(\vec{q}_\perp, y) = \int \frac{dx d^2 k_\perp}{x(1-x)} \frac{\Gamma_B(\vec{k}_\perp, x)}{M_B^2 - M_o^2}$$

$$\times\left[K^{(3)}(q_\perp,y;\vec{k}_\perp,x)+K^{(4)}(\vec{q}_\perp,y;\vec{k}_\perp,x)\right.$$
$$\left.+K_\times^{(4)}(\vec{q}_\perp,y;\vec{k}_\perp,x)+K_\times^{(4)anti}(\vec{q}_\perp,y;\vec{k}_\perp,x)\right], \tag{40}$$

where

$$K_\times^{(4)}=\left(\frac{g^2}{16\pi^3}\right)^2\int\frac{d^2p_\perp dz\theta(1-z)}{z(x-z)(1-z-x-y)}\left(K'_\times+K''_\times\right), \tag{41}$$

with

$$\begin{aligned}K'_\times=&\frac{\theta(z-y)\theta(x-z)}{(z-y)[M_B^2-\frac{p_\perp^2+m^2-i\varepsilon}{z}-\frac{k_\perp^2+m^2-i\varepsilon}{1-x}-\frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{x-z}]}\\&\times\frac{1}{M_B^2-\frac{(p-q)_\perp^2+m_\sigma^2-i\varepsilon}{z-y}-\frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{x-z}-\frac{k_\perp^2+m^2-i\varepsilon}{1-x}-\frac{q_\perp^2+m^2-i\varepsilon}{y}}\\&\times\frac{1}{M_B^2-\frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{x-z}-\frac{(p-k-q)_\perp^2+m^2-i\varepsilon}{1-z-x-y}-\frac{q_\perp^2+m_\sigma^2-i\varepsilon}{y}}\\&+[x\leftrightarrow y,k_\perp\leftrightarrow q_\perp],\end{aligned} \tag{42}$$

and

$$\begin{aligned}K''_\times=&\frac{\theta(y-z)\theta(x-z)}{(y-z)}\\&\times\frac{1}{M_B^2-\frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{x-z}-\frac{(1-p-k-q)_\perp^2+m^2-i\varepsilon}{1-z-x-y}-\frac{p_\perp^2+m^2-i\varepsilon}{z}-\frac{(q-p)_\perp^2+m_\sigma^2-i\varepsilon}{y-z}}\\&\times\frac{1}{M_B^2-\frac{(q-p)_\perp^2+m_\sigma^2-i\varepsilon}{y-z}-\frac{(1-p-k-q)_\perp^2+m^2-i\varepsilon}{1-z-x-y}-\frac{k_\perp^2+m^2-i\varepsilon}{x}}\\&\times\left\{\frac{1}{M_B^2-\frac{(k-p)_\perp^2+m_\sigma^2-i\varepsilon}{x-z}-\frac{(1-p-k-q)_\perp^2+m^2-i\varepsilon}{1-z-x-y}-\frac{q_\perp^2+m_\sigma^2-i\varepsilon}{y}}\right.\\&\left.+\frac{1}{M_B^2-\frac{(q-p)_\perp^2+m_\sigma^2-i\varepsilon}{y-z}-\frac{q_\perp^2+m^2-i\varepsilon}{1-y}-\frac{p_\perp^2+m^2-i\varepsilon}{z}}\right\}\\&+[x\leftrightarrow 1-x;z\leftrightarrow 1+z-x-y,p_\perp\leftrightarrow p_\perp\leftrightarrow(1+p-k-q)_\perp;y\leftrightarrow 1-y].\end{aligned} \tag{43}$$

The contribution of the pair particle and anti-particle to the kernel of (40) is given by:

$$\Gamma_B(q_\perp,y)=\int\frac{dxd^2k_\perp}{x(1-x)}\frac{\Gamma_B(k_\perp,x)}{M_B^2-M_o^2}\left\{K^{(3)}(q_\perp,y;k_\perp,x)+K^{(4)}(q_\perp,y;k_\perp,x)\right.$$
$$\left.+K_\times^{(4)}(q_\perp,y;k_\perp,x)+K_\times^{(4)ant}(q_\perp,y;k_\perp,x)\right\}, \tag{44}$$

where the kernel for anti-particle is given by:

$$K_{\times}^{(4)ant} = \left(\frac{g^2}{16\pi^3}\right)^2 \int \frac{d^2k_\perp dz\theta(1+z)\theta(-z)}{|z|(y+|z|)(1-|z|-x-y)(x+|z|)}$$
$$\times \frac{\theta(y+z)\theta(x+z)}{M^2 - \frac{w_\perp^2+m^2-i\varepsilon}{y} - \frac{(K+k-p-w)_\perp^2+m^2-i\varepsilon}{1-|z|-x-y} + \frac{k_\perp^2+m^2-i\varepsilon}{|z|} - \frac{p_\perp^2+m_\sigma^2-i\varepsilon}{x}}$$
$$\times \frac{1}{M^2 - \frac{(p-k)_\perp^2+m_\sigma^2-i\varepsilon}{x+|z|} - \frac{(K+k-p-w)_\perp^2+m^2-i\varepsilon}{1-|z|-x-y} - \frac{p_\perp^2+m_\sigma^2-i\varepsilon}{x}}$$
$$\times \frac{1}{M^2 - \frac{(w-k)_\perp^2+m_\sigma^2-i\varepsilon}{y+|z|} - \frac{(K+k-p-w)_\perp^2+m^2-i\varepsilon}{1-|z|-x-y} - \frac{w_\perp^2+m_\sigma^2-i\varepsilon}{y}}. \tag{45}$$

Equation (40) is the Bethe-Salpeter equation with the kernel expanded up to fourth order in g in the null-plane in the cross-ladder approximation.

10 Conclusions

The Bethe-Salpeter equation is constructed via few-body propagation, it has been demonstrated that the ladder diagram in the light-front gives an important contribution to the covariance of the process (equal-time three and four particles) [7, 8]. In our case, we are observing that for the Bethe-Salpeter construction, besides the equal-time three- and four particles we also should have a non-vanishing contribution due to pair production. So we expect and suspect that the crossed ladder diagram in such a construction will also give a non-vanishing relevant contribution; which contribution is of the z-diagram type (backward-moving particles).

We have determined algebraically that there must be an inclusion of the antiparticle contribution to the Bethe-Salpeter equation for bound states. Our work therefore makes it clear that for the Bethe-Salpeter equation for two propagating bosons there is a nonvanishing contribution from antiparticles besides the contribution from particles. Of course, the amount by which this contribution (numerical result) affects the overall result is yet to be checked.

Acknowledgements J.H.O. Sales thanks FAPESP (Proc. 2007/06527-0) and FAPEMIG (APQ-02547-08) for their partial financial support and the hospitality of Instituto de Física Teórica/UNESP, where part of this work has been done.

References

1. Jackiw, R.: Ann. Phys. **48**, 292 (1968)
2. Sawicki, M.: Phys. Rev. D **44**, 433 (1991)
3. Sawicki, M.: Phys. Rev. D **46**, 474 (1992)
4. Carbonell, J., Karmanov, V.A.: hep-th/0505262
5. Karmanov, V.A., Carbonell, J.: nucl-th/0510051
6. Frederico, T., Sales, J.H.O., Carlson, B.V., Sauer, P.U.: Few-Body Syst. **10**, 123 (1999)
7. Sales, J.H., Frederico, T., Carlson, B.V., Sauer, P.U.: Phys. Rev. C **61**, 044003 (2000)
8. Sales, J.H., Frederico, T., Carlson, B.V., Sauer, P.U.: Phys. Rev. C **63**, 064003 (2001)
9. Brodsky, S.J., Pauli, H.-C., Pinski, S.: Phys. Rep. **301**, 299 (1998)
10. Frankfurt, L.L., Strikman, M.I.: Nucl. Phys. B **148**, 107 (1979)
11. Frankfurt, L.L., Strikman, M.I.: Phys. Rep. **76**, 215 (1981)

12. Frankfurt, L.L., Frederico, T., Strikmam, M.I.: Phys. Rev. C **48**, 2182 (1993)
13. Frederico, T., Miller, G.A.: Phys. Rev. D **45**, 4207 (1992)
14. Frederico, T., Miller, G.A.: Phys. Rev. D **50**, 210 (1994)
15. Carbonell, J., Desplanques, B., Karmanov, V., Mathiot, J.F.: Phys. Rep. **300**, 215 (1998)
16. Weinberg, S.: Phys. Rev. **150**, 1313 (1966)
17. Ji, C.-R., Furnstahl, R.J.: Phys. Lett. B **167**, 11 (1986)
18. Ji, C.-R.: Phys. Lett. B **167**, 16 (1986)
19. Ji, C.-R.: Phys. Lett. B **322**, 389 (1994)
20. Glasek, S., Harindranath, A., Pinsky, S., Shigemitsu, J., Wilson, K.G.: Phys. Rev. D **47**, 1599 (1993)
21. Perry, R.J., Harindranath, A., Wilson, K.G.: Phys. Rev. Lett. **65**, 2959 (1990)
22. Brodsky, S.J., Ji, C.R., Sawicki, M.: Phys. Rev. D **32**, 1530 (1985)
23. Drell, S.D., Levy, D.J., Yan, J.-M.: Phys. Rev. D **1**, 1035 (1970)
24. Kogut, J.B., Soper, D.E.: Phys. Rev. D **1**, 2901 (1970)
25. Cooke, J.R., Miller, G.A.: Phys. Rev. C **62**, 054008 (2000)
26. Messiah, A.: Quantum Mechanics, vol. II. North-Holland, Amsterdam (1962)
27. Dirac, P.A.M.: Rev. Mod. Phys. **21**, 392 (1949)
28. Namyslowski, J.M.: Prog. Part. Nucl. Phys. **14**, 49 (1985)